*Summer Review Packet for Students Entering AP Calculus BC*

**Complex Fractions**

**When simplifying complex fractions, multiply by a fraction equal to 1 which has a numerator and denominator composed of the common denominator of all the denominators in the complex fraction.**

**Example:**

$$\frac{-7-\frac{6}{x+1}}{\frac{5}{x+1}} = \frac{-7-\frac{6}{x+1}}{\frac{5}{x+1}} \bullet \frac{x+1}{x+1} = \frac{-7x-7-6}{5} = \frac{-7x-13}{5}$$

$$\frac{\frac{-2}{x}+\frac{3x}{x-4}}{5-\frac{1}{x-4}} = \frac{\frac{-2}{x}+\frac{3x}{x-4}}{5-\frac{1}{x-4}} \bullet \frac{x(x-4)}{x(x-4)} = \frac{-2(x-4)+3x(x)}{5(x)(x-4)-1(x)} = \frac{-2x+8+3x^2}{5x^2-20x-x} = \frac{3x^2-2x+8}{5x^2-21x}$$

**Simplify each of the following.**

1. $\dfrac{\frac{25}{a}-a}{5+a}$

2. $\dfrac{2-\frac{4}{x+2}}{5+\frac{10}{x+2}}$

3. $\dfrac{4-\frac{12}{2x-3}}{5+\frac{15}{2x-3}}$

4. $\dfrac{\frac{x}{x+1}-\frac{1}{x}}{\frac{x}{x+1}+\frac{1}{x}}$

5. $\dfrac{1-\frac{2x}{3x-4}}{x+\frac{32}{3x-4}}$

## Functions

**To evaluate a function for a given value, simply plug the value into the function for x.**

**Recall:** $\left(f \circ g\right)(x) = f(g(x))$ *OR* $f[g(x)]$ read "***f*** of ***g*** of ***x***" Means to plug the inside function (in this case g(x) ) in for x in the outside function (in this case, f(x)).

**Example:** Given $f(x) = 2x^2 + 1$ *and* $g(x) = x - 4$ find *f(g(x))*.

$$\begin{aligned} f(g(x)) &= f(x-4) \\ &= 2(x-4)^2 + 1 \\ &= 2(x^2 - 8x + 16) + 1 \\ &= 2x^2 - 16x + 32 + 1 \\ f(g(x)) &= 2x^2 - 16x + 33 \end{aligned}$$

**Let** $f(x) = 2x + 1$ *and* $g(x) = 2x^2 - 1$**. Find each.**

6. $f(2) =$ ____________ 7. $g(-3) =$ ____________ 8. $f(t+1) =$ ____________

9. $f\left[g(-2)\right] =$ ____________ 10. $g\left[f(m+2)\right] =$ ____________ 11. $\dfrac{f(x+h) - f(x)}{h} =$ ____________

**Let** $f(x) = \sin x$ **Find each exactly.**

12. $f\left(\dfrac{\pi}{2}\right) =$ ____________ 13. $f\left(\dfrac{2\pi}{3}\right) =$ ____________

**Let** $f(x) = x^2$, $g(x) = 2x + 5$, *and* $h(x) = x^2 - 1$**. Find each.**

14. $h\left[f(-2)\right] =$ _______ 15. $f\left[g(x-1)\right] =$ _______ 16. $g\left[h(x^3)\right] =$ _______

**Find $\frac{f(x+h)-f(x)}{h}$ for the given function *f*.**

17. $f(x)=9x+3$

18. $f(x)=5-2x$

19. $f(x)=x^2+1$

20. $f(x)=x^3$

## Intercepts and Points of Intersection

To find the x-intercepts, let y = 0 in your equation and solve.
To find the y-intercepts, let x = 0 in your equation and solve.

**Example:** $y=x^2-2x-3$

$x-\text{int. } (Let\ y=0)$

$0=x^2-2x-3$

$0=(x-3)(x+1)$

$x=-1 \ or \ x=3$

$x-\text{intercepts } (-1,0) \ and \ (3,0)$

$y-\text{int. } (Let\ x=0)$

$y=0^2-2(0)-3$

$y=-3$

$y-\text{intercept } (0,-3)$

**Find the x and y intercepts for each.**

21. $y=2x-5$

22. $y=x^2+x-2$

23. $y^2 = x^3 - 4x$

**Use substitution or elimination method to solve the system of equations.**
**Example:**

$$x^2 + y - 16x + 39 = 0$$
$$x^2 - y^2 - 9 = 0$$

Elimination Method

$2x^2 - 16x + 30 = 0$

$x^2 - 8x + 15 = 0$

$(x-3)(x-5) = 0$

$x = 3 \text{ and } x = 5$

Plug $x = 3$ *and* $x = 5$ into one original

$3^2 - y^2 - 9 = 0$ $\quad$ $5^2 - y^2 - 9 = 0$

$-y^2 = 0$ $\quad$ $16 = y^2$

$y = 0$ $\quad$ $y = \pm 4$

Points of Intersection $(5,4)$, $(5,-4)$ *and* $(3,0)$

Substitution Method

Solve one equation for one variable.

$y^2 = -x^2 + 16x - 39$ (1st equation solved for y)

$x^2 - (-x^2 + 16x - 39) - 9 = 0$ Plug what $y^2$ is equal to into second equation.

$2x^2 - 16x + 30 = 0$ (*The rest is the same as previous example*)

$x^2 - 8x + 15 = 0$

$(x-3)(x-5) = 0$

$x = 3 \text{ or } x - 5$

**Find the point(s) of intersection of the graphs for the given equations.**

24. $x + y = 8$
$4x - y = 7$

25. $x^2 + y = 6$
$x + y = 4$

## Interval Notation

26. Complete the table with the appropriate notation or graph.

| Solution | Interval Notation | Graph |
|---|---|---|
| $-2 < x \le 4$ | | |
| | $[-1, 7)$ | |
| | |  |

Solve each equation. State your answer in BOTH interval notation and graphically.

27. $2x - 1 \ge 0$

28. $-4 \le 2x - 3 < 4$

29. $\frac{x}{2} - \frac{x}{3} > 5$

## Domain and Range

**Find the domain and range of each function. Write your answer in INTERVAL notation.**

30. $f(x) = x^2 - 5$

31. $f(x) = -\sqrt{x+3}$

32. $f(x) = 3\sin x$

33. $f(x) = \frac{2}{x-1}$

## Inverses

To find the inverse of a function, simply switch the x and the y and solve for the new "y" value.
**Example:**

| | |
|---|---|
| $f(x)=\sqrt[3]{x+1}$ | Rewrite f(x) as y |
| $y=\sqrt[3]{x+1}$ | Switch x and y |
| $x=\sqrt[3]{y+1}$ | Solve for your new y |
| $(x)^3=\left(\sqrt[3]{y+1}\right)^3$ | Cube both sides |
| $x^3=y+1$ | Simplify |
| $y=x^3-1$ | Solve for y |
| $f^{-1}(x)=x^3-1$ | Rewrite in inverse notation |

**Find the inverse for each function.**

**34.** $f(x)=2x+1$

**35.** $f(x)=\dfrac{x^2}{3}$

Also, recall that to PROVE one function is an inverse of another function, you need to show that:
$f(g(x))=g(f(x))=x$

**Example:**

**If:** $f(x)=\dfrac{x-9}{4}$ *and* $g(x)=4x+9$ **show *f(x) and g(x)* are inverses of each other.**

$$g(f(x))=4\left(\frac{x-9}{4}\right)+9$$
$$=x-9+9$$
$$=x$$

$$f(g(x))=\frac{(4x+9)-9}{4}$$
$$=\frac{4x+9-9}{4}$$
$$=\frac{4x}{4}$$
$$=x$$

$f(g(x))=g(f(x))=x$ therefore they are inverses of each other.

**Prove *f and g* are inverses of each other.**

**36.** $f(x)=\frac{x^3}{2}$ $\quad g(x)=\sqrt[3]{2x}$

**37.** $f(x)=9-x^2,\ x\geq 0$ $\quad g(x)=\sqrt{9-x}$

## <u>Equation of a line</u>

**Slope intercept form:** $y=mx+b$

**Point-slope form:** $y-y_1=m(x-x_1)$

**Vertical line:** x = c (slope is undefined)

**Horizontal line:** y = c (slope is 0)

38. Use slope-intercept form to find the equation of the line having a slope of 3 and a y-intercept of 5.

39. Determine the equation of a line passing through the point (5, -3) with an undefined slope.

40. Determine the equation of a line passing through the point (-4, 2) with a slope of 0.

41. Use point-slope form to find the equation of the line passing through the point (0, 5) with a slope of 2/3.

42. Find the equation of a line passing through the point (2, 8) and parallel to the line $y = \frac{5}{6}x - 1$.

43. Find the equation of a line perpendicular to the y- axis passing through the point (4, 7).

44. Find the equation of a line passing through the points (-3, 6) and (1, 2).

45. Find the equation of a line with an x-intercept (2, 0) and a y-intercept (0, 3).

**<u>Radian and Degree Measure</u>**

| | |
|---|---|
| Use $\frac{180^\circ}{\pi \text{ } radians}$ to get rid of radians and convert to degrees. | Use $\frac{\pi \text{ } radians}{180^\circ}$ to get rid of degrees and convert to radians. |

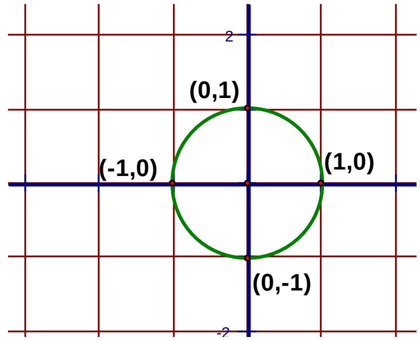
46. Convert to degrees: a. $\frac{5\pi}{6}$ b. $\frac{4\pi}{5}$ c. 2.63 radians

47. Convert to radians: a. $45^\circ$ b. $-17^\circ$ c. $237^\circ$

## Unit Circle

You can determine the sine or cosine of a quadrantal angle by using the unit circle. The x-coordinate of the circle is the cosine and the y-coordinate is the sine of the angle.

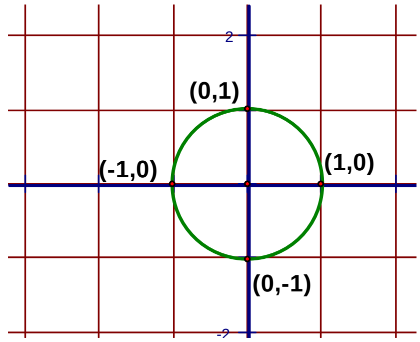
**Example:** $\sin 90^\circ = 1$ $\cos\frac{\pi}{2} = 0$



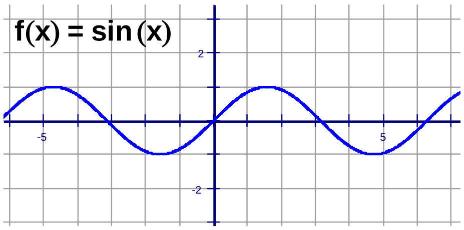

48. $a.)\ \sin 180^\circ$ $b.)\ \cos 270^\circ$

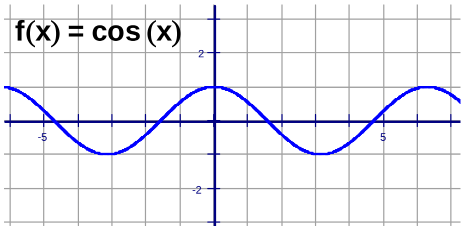
$c.)\ \sin(-90^\circ)$ $d.)\ \sin \pi$

$e.)\ \cos 360^\circ$ $f.)\ \cos(-\pi)$




## Graphing Trig Functions




y = sin x and y = cos x have a period of $2\pi$ and an amplitude of 1. Use the parent graphs above to help you sketch a graph of the functions below. For $f(x) = A\sin(Bx + C) + K$, A = amplitude, $\frac{2\pi}{B}$ = period, $\frac{C}{B}$ = phase shift (positive C/B shift left, negative C/B shift right) and K = vertical shift.

**Graph two complete periods of the function.**

49. $f(x) = 5\sin x$

50. $f(x) = \sin 2x$

51. $f(x) = -\cos\left(x - \frac{\pi}{4}\right)$

52. $f(x) = \cos x - 3$

**Trigonometric Equations:**

Solve each of the equations for $0 \le x < 2\pi$. Isolate the variable, sketch a reference triangle, find all the solutions within the given domain, $0 \le x < 2\pi$. Remember to double the domain when solving for a double angle. Use trig identities, if needed, to rewrite the trig functions. (See formula sheet at the end of the packet.)

53. $\sin x = -\frac{1}{2}$

54. $2\cos x = \sqrt{3}$

55. $\cos 2x = \frac{1}{\sqrt{2}}$

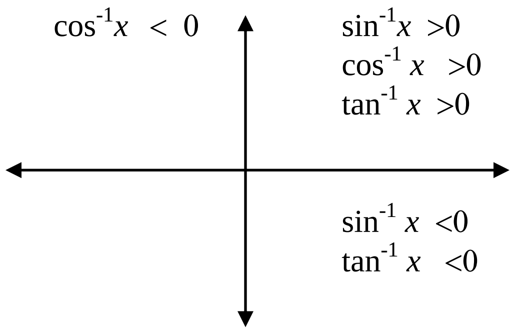
56. $4\cos^2 x - 3 = 0$

## Inverse Trigonometric Functions:

**Recall:** Inverse Trig Functions can be written in one of ways:

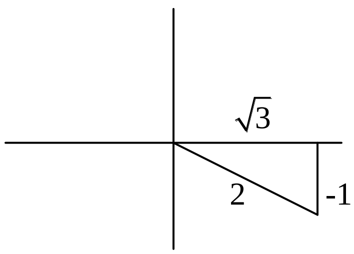
$$\arcsin(x) \qquad\qquad \sin^{-1}(x)$$

Inverse trig functions are defined only in the quadrants as indicated below due to their restricted domains.




**Example:**
Express the value of "y" in radians.

$y = \arctan \dfrac{-1}{\sqrt{3}}$ Draw a reference triangle.




This means the reference angle is 30° or $\frac{\pi}{6}$. So, y = – $\frac{\pi}{6}$ so that it falls in the interval from

$$\frac{-\pi}{2} < y < \frac{\pi}{2} \qquad\qquad \text{Answer: } y = -\frac{\pi}{6}$$

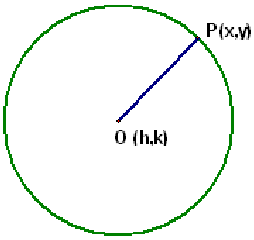
**For each of the following, express the value for "y" in radians.**

57. $y = \arcsin \dfrac{-\sqrt{3}}{2}$

58. $y = \arccos(-1)$

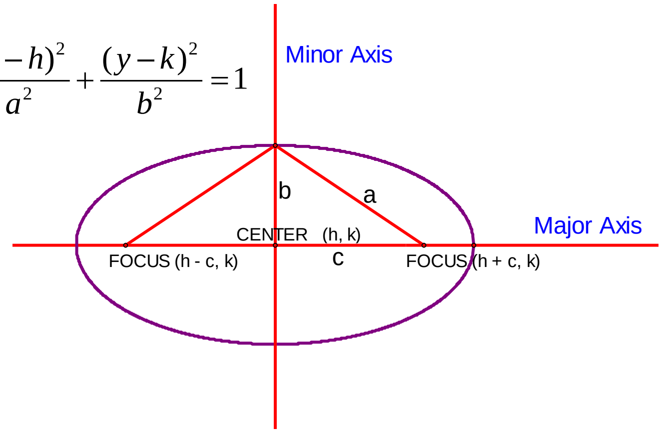
59. $y = \arctan(-1)$

## <u>Circles and Ellipses</u>




$$r^2 = (x-h)^2 + (y-k)^2$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

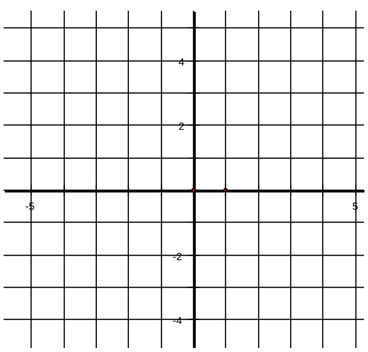



For a circle centered at the origin, the equation is $x^2 + y^2 = r^2$, where **r** is the radius of the circle.

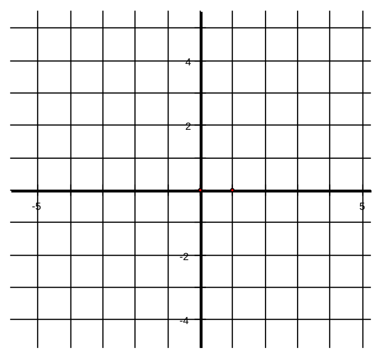
For an ellipse centered at the origin, the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where **a** is the distance from the center to the ellipse along the x-axis and **b** is the distance from the center to the ellipse along the y-axis. If the larger number is under the $y^2$ term, the ellipse is elongated along the *y*-axis. For our purposes in Calculus, you will not need to locate the foci.

**Graph the circles and ellipses below:**

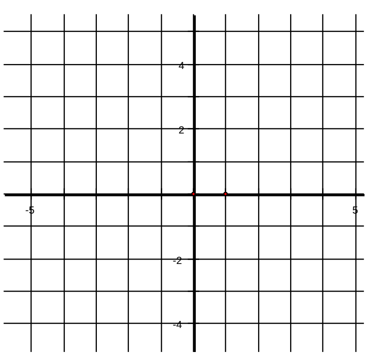
60. $x^2 + y^2 = 16$



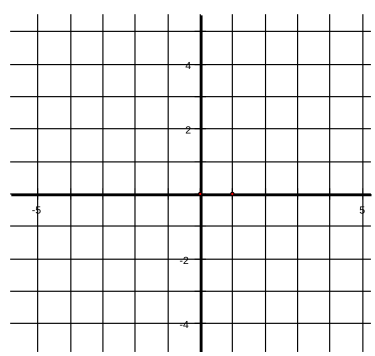
61. $x^2 + y^2 = 5$



62. $\frac{x^2}{1} + \frac{y^2}{9} = 1$



63. $\frac{x^2}{16} + \frac{y^2}{4} = 1$



## Limits

**Finding limits numerically.**

Complete the table and use the result to estimate the limit. If the limit is increasing without bound, write $\infty$ . If the limit is decreasing without bound, write $-\infty$ .

64. $\lim_{x \to 4} \frac{x-4}{x^2-3x-4}$

| x | 3.9 | 3.99 | 3.999 | 4.001 | 4.01 | 4.1 |
|---|---|---|---|---|---|---|
| f(x) | | | | | | |

65. $\lim_{x \to -5} \frac{\sqrt{4-x}-3}{x+5}$

| x | -5.1 | -5.01 | -5.001 | -4.999 | -4.99 | -4.9 |
|---|---|---|---|---|---|---|
| f(x) | | | | | | |

**Finding limits graphically.**

Find each limit graphically. Use your calculator to assist in graphing. If the limit is increasing without bound, write $\infty$ . If the limit is decreasing without bound, write $-\infty$ .

66. $\lim_{x \to 0} \cos x$

67. $\lim_{x \to 5} \frac{2}{x-5}$

68. $\lim_{x \to 1} f(x)$

$$f(x) = \begin{cases} x^2+3, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

## Evaluating Limits Analytically

Solve by direct substitution whenever possible. If needed, rearrange the expression so that you can do direct substitution. (NO CALCULATORS)

69. $\lim_{x \to 2}(4x^2+3)$

70. $\lim_{x \to 1} \frac{x^2+x+2}{x+1}$

71. $\lim_{x \to 0} \sqrt{x^2 + 4}$

72. $\lim_{x \to \pi} \cos x$

73. $\lim_{x \to 1} \left( \frac{x^2 - 1}{x - 1} \right)$

74. $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$

75. $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$ HINT: Rationalize the numerator.

76. $\lim_{x \to 3} \frac{3 - x}{x^2 - 9}$

77. $\lim_{h \to 0} \frac{2(x+h) - 2x}{h}$

**One-Sided Limits**

Find the limit if it exists. First, try to solve for the overall limit. If an overall limit exists, then the one-sided limit will be the same as the overall limit. If not, investigate the one-sided limit using a test value close to (within .5 unit) of the value *x* is approaching.

**Evaluate the limit. No Calculator**

$\lim\limits_{x\to4^+}\frac{3x}{x-4}$

- Check the limit by plugging in x = 4: $\lim\limits_{x\to4^+}\frac{3x}{x-4}=\frac{3(4)}{4-4}=\frac{12}{0}$ .

- Since we get $\frac{\text{not }0}{0}$ when plugging in 4, $\lim\limits_{x\to4^+}\frac{3x}{x-4}$ will be $\infty$ or $-\infty$. To determine the sign of the infinity, plug a number in that is to the right of 4 (greater than 4 - but not greater than 4.5). The sign of this answer will be the sign of the infinity.

$\frac{3x}{x-4}\to\frac{3(4.5)}{4.5-4}=\frac{13.5}{.5}=27$

Since we got a positive answer when we plugged in a number to the right of 4 (greater than 4), the limit is $\infty$ .

$\lim\limits_{x\to4^+}\frac{3x}{x-4}=\infty$

78. $\lim\limits_{x\to5^+}\frac{x-5}{x^2-25}$

79. $\lim\limits_{x\to-3^-}\frac{x}{\sqrt{x^2-9}}$

80. $\lim\limits_{x\to10^+}\frac{|x-10|}{x-10}$

81. $\lim\limits_{x\to5^-}\left(-\frac{3}{x+5}\right)$

## Vertical Asymptotes

Determine the vertical asymptotes for the function. Set the denominator equal to zero to find the x-value for which the function is undefined. That will be the vertical asymptote.

82. $f(x)=\frac{1}{x^2}$

83. $f(x)=\frac{x^2}{x^2-4}$

84. $f(x)=\frac{2+x}{x^2(1-x)}$

## Horizontal Asymptotes

To find the horizontal asymptotes of a function, investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Limits at infinity can be classified as one of three cases.

**Case I**. Degree of the numerator is less than the degree of the denominator (the function is bottom heavy). $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to-\infty} f(x)=0$. The horizontal asymptote is y = 0.

**Case II.** Degree of the numerator is the same as the degree of the denominator. The limits and the horizontal asymptote are all equal to the ratio of the lead coefficients.

**Case III**. Degree of the numerator is greater than the degree of the denominator (top heavy). The limit is either $\infty$ (increases without bound) or $-\infty$ (decreases without bound). This can be determined by the exponent of the quotient of the degrees in the numerator and denominator. If the exponent of the quotient of the degrees in the numerator and denominator is odd, then $\lim_{x\to-\infty} f(x)=-\infty$ and $\lim_{x\to\infty} f(x)=\infty$. If the exponent of the quotient of the degree is even, then $\lim_{x\to-\infty} f(x)=\infty$ and $\lim_{x\to\infty} f(x)=\infty$. There is no horizontal asymptote. (If the degree of the numerator is exactly 1 more than the degree of the denominator, then there exists a slant asymptote, which is determined by long division or synthetic division.)

**Determine all Horizontal Asymptotes. Check your answer with the graphing calculator.**

85. $f(x)=\frac{x^2-2x+1}{x^3+x-7}$

86. $f(x)=\frac{5x^3-2x^2+8}{4x-3x^3+5}$

87. $f(x)=\frac{4x^5}{x^2-7}$

**Determine each limit as x goes to infinity.**

88. $\lim_{x\to\infty}\left(\frac{2x-5+4x^2}{3-5x+x^2}\right)$

89. $\lim_{x\to\infty}\left(\frac{2x-5}{3-5x+3x^2}\right)$

90. $\lim_{x\to\infty}\left(\frac{7x+6-2x^3}{3+14x+x^2}\right)$

## Limits to Infinity

A rational function does not have a limit if it goes to $\pm\infty$, however, you can state the direction the limit is headed if both the left and right hand side go in the same direction.

Determine each limit if it exists. If the limit approaches $\infty$ *or* $-\infty$, please state which one the limit approaches.

91. $\lim_{x\to -1^+} \frac{1}{x+1} =$

92. $\lim_{x\to 1^+} \frac{2+x}{1-x} =$

93. $\lim_{x\to 0} \frac{2}{\sin x} =$

**<u>Continuity</u>**

A continuous function can be drawn without lifting your pencil. In order for a function, $f(x)$, to be continuous at $x = c$, these three conditions must be met:

1.) $f(\text{c})$ exists.
2.) $\lim_{x\to c} f(x)$ exists.
3.) $\lim_{x\to c} f(x) = f(c)$

**Continuity of Function Families:**

<u>Polynomials</u> – continuous everywhere on $(-\infty,\infty)$.
<u>Rational Functions</u> – continuous everywhere except for values of *x* that make the denominator zero.
<u>Piecewise Functions</u> – check the continuity at the endpoints of each branch.
<u>Trig Functions</u> – sine and cosine are continuous everywhere; tangent and secant have discontinuities at all odd multiples of $\frac{\pi}{2}$; cosecant and cotangent have discontinuities at all multiples of $\pi$.

**Removable Discontinuities**

If a function, $f(x)$, has a discontinuity at $x = c$ but $\lim_{x\to c} f(x)$ exists, then $f(x)$ has a removable discontinuity at $x = c$.

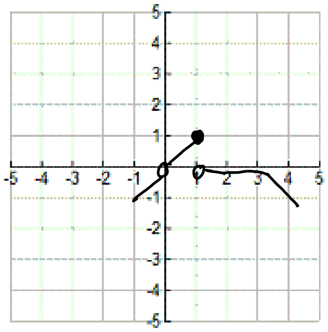
Determine the values of *x*, if any, for which the function is not continuous. Determine if the discontinuity is removable.

94. $f(x) = \begin{cases} |x-3| & , \quad x \neq 3 \\ 0 & , \quad x = 3 \end{cases}$

95. $f(x) = (x-2)^5$

96. $f(x) = \frac{x}{x^2-1}$

97.




98. $f(x)=\begin{cases} |x-3| & , \quad x \neq 3 \\ 2 & , \quad x = 3 \end{cases}$

99. $f(x)=\dfrac{(2x+3)(x-1)}{x-1}$

100. $f(x)=\begin{cases} \dfrac{x^2-4x+3}{x-3} & , \quad x \neq 3 \\ 5 & , \quad x = 3 \end{cases}$

# Formula Sheet (should be memorized)

Reciprocal Identities: $\csc x = \dfrac{1}{\sin x}$ $\sec x = \dfrac{1}{\cos x}$ $\cot x = \dfrac{1}{\tan x}$

Quotient Identities: $\tan x = \dfrac{\sin x}{\cos x}$ $\cot x = \dfrac{\cos x}{\sin x}$

Pythagorean Identities: $\sin^2 x + \cos^2 x = 1$ $\tan^2 x + 1 = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$

Double Angle Identities: $\sin 2x = 2\sin x \cos x$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\\ &= 2\cos^2 x - 1\end{aligned}$$

Logarithms: $y = \log_a x$ is equivalent to $x = a^y$

Product property: $\log_b mn = \log_b m + \log_b n$

Quotient property: $\log_b \dfrac{m}{n} = \log_b m - \log_b n$

Power property: $\log_b m^p = p \log_b m$

Property of equality: If $\log_b m = \log_b n$, then m = n

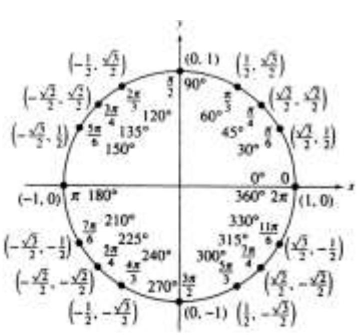
Change of base formula: $\log_a n = \dfrac{\log_b n}{\log_b a}$

Derivative of a Function: Slope of a tangent line to a curve or the derivative: $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Slope-intercept form: $y = mx + b$

Point-slope form: $y - y_1 = m(x - x_1)$

Standard form: Ax + By + C = 0




**You should also be able to do the following without hesitation:**

**SOLVE – any equation (i.e. linear, quadratic, exponential, logarithmic, trigonometric)**

**FACTOR – by all methods (i.e. difference of two squares, sum/difference of cubes, trinomials, perfect square trinomials)**

**DIVIDE polynomials using synthetic or long division**

**USE graphing calculator to graph functions, find intersection points, zeros, and values**

## **Graphs You Should Know**